On the Calculation of the Co-factors of Alternants of the Fourth Order.

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1. If we subtract the first row of an alternant of the n^{th} order in which the first column is a column of units, from each of the other rows, the determinant is reduced to the order $n-1$, and each row may be divided by one of the differences involving the first letter. For example,

$$\begin{vmatrix} 1, & a^2, & a^4, & a^5 \\ 1, & b^2, & b^4, & b^5 \\ 1, & c^2, & c^4, & c^5 \\ 1, & d^2, & d^4, & d^5 \end{vmatrix}$$

$$=(b-a)(c-a)(d-a)\begin{vmatrix} b+a, & b^3+b^2a+ba^2+a^3, & b^4+b^3a+b^2a^2+ba^3+a^4 \\ c+a, & c^3+c^2a+ca^2+a^3, & c^4+c^3a+c^2a^2+ca^3+a^4 \\ d+a, & d^3+d^2a+da^2+a^3, & d^4+d^3a+d^2a^2+da^3+a^4 \end{vmatrix}.$$

Subtracting the second column multiplied by a from the third, then the first column multiplied by a^2 from the second, and decomposing the result by columns, the determinant in the right hand member becomes

$$\begin{vmatrix} b+a, & b^3+b^2a, & b^4 \\ c+a, & c^3+c^2a, & c^4 \\ d+a, & d^3+d^2a, & d^4 \end{vmatrix} = a^2A(0,\ 2,\ 4)+aA(0,\ 3,\ 4)+aA(1,\ 2,\ 4)+A(1,\ 3,\ 4),$$

in which the alternants involve the letters b, c and d. These alternants are divisible by $\zeta^{\frac{1}{2}}(b,\ c,\ d)$; hence, since

$$\zeta^{\frac{1}{2}}(a,\ b,\ c,\ d)=(b-a)(c-a)(d-a)\,\zeta^{\frac{1}{2}}(b,\ c,\ d),$$

we have, denoting the co-factor of an alternant by the symbol α,

$$\alpha(0,2,4,5)=a^2\alpha(0,\ 2,\ 4)+a\alpha(0,\ 3,\ 4)+a\alpha(1,\ 2,\ 4)+\alpha(1,3,4).$$

2. It is obvious that the process is general, the numbers of partial columns in the columns of the determinant of the $(n-1)^{\text{th}}$ order being the differences between the consecutive exponents of the original alternant, so that the whole number of alternants of the $(n-1)^{\text{th}}$ order is the product of these differences.

In particular when $n=4$, $\alpha(0,p,q,r)$ is a symmetric function of a, b, c and d of the degree $p+q+r-6$, and we have

$$(1)\qquad \alpha(0,p,q,r)=\sum_u\sum_v\sum_w a^\theta\alpha(u,\ v,\ w),\qquad \left\{\begin{matrix} u=0,\ 1,\ 2,\ \ldots p-1 \\ v=p,\ p+1,\ \ldots q-1 \\ w=q,\ q+1,\ \ldots r-1 \end{matrix}\right\};$$

the degree of $\alpha(u, v, w)$ is $u+v+w-3$, and $\theta = p+q+r-(u+v+w)-3$; the greatest value of θ being $r-3$, and its least value zero.

The terms of this sum may be imagined to be arranged in a solid rectangular block, analogous to the plane rectangular array of binomials in §3 of the preceding article. Let us suppose the first symbol, u, of $\alpha(u, v, w)$ to increase downward from zero in the top face to $p-1$ in the bottom face; the second, v, to increase from p on the left to $q-1$ on the right, and the third, w, to increase from q in the front face to $r-1$ in the back face. The first term $a^{r-3}\alpha(0, p, q)$ will thus occupy the upper, left-hand, front corner, and $\alpha(p-1, q-1, r-1)$ will occupy the lower, right-hand, back corner.

3. Now if we remove the terms in the top face, in each of which $u=0$, all the rest may be divided by bcd by subtracting unity from each symbol; and if we also remove the back face, which contains the only term in which $\theta = 0$, all the rest may be divided by a. Dividing the remaining block of terms by $abcd$, the first symbol of the α's will run from zero to $p-2$, the second from $p-1$ to $q-2$, and the third from $q-1$ to $r-3$; the term at the upper, left-hand, front corner will be $a^{r-5}\alpha(0, p-1, q-1)$, and the term at the lower, right-hand, back corner will be $\alpha(p-2, q-2, r-3)$. In other words, the remaining block of terms after division is the value of $\alpha(0, p-1, q-1, r-2)$. Denoting by Q the sum of the terms removed, we have therefore

$$\alpha(0, p, q, r) = Q + abcd.\alpha(0, p-1, q-1, r-2), \tag{2}$$

from which it appears that Q is a symmetric function of a, b, c and d.

4. The function Q consists of the terms situated in the top face and in the back face of the rectangular block. Imagining the top face to be rotated about the common edge into the plane of the back face, the terms constituting Q may be written thus:

$$\begin{aligned}
Q = {} & a^{r-3}\alpha(0, p, q) + a^{r-4}\alpha(0, p+1, q) + \ldots + a^{r+p-q-2}\alpha(0, q-1, q) \\
& + a^{r-4}\alpha(0, p, q+1) + a^{r-5}\alpha(0, p+1, q+1) + \ldots + a^{r+p-q-3}\alpha(0, q-1, q+1) \\
& \quad \vdots \qquad\qquad\qquad\qquad \vdots \\
& + a^{q-2}\alpha(0, p, r-1) + a^{q-3}\alpha(0, p+1, r-1) + \ldots + a^{p-1}\alpha(0, q-1, r-1) \\
& + a^{q-3}\alpha(1, p, r-1) + a^{q-4}\alpha(1, p+1, r-1) + \ldots + a^{p-2}\alpha(1, q-1, r-1) \\
& \quad \vdots \qquad\qquad\qquad\qquad \vdots \\
& + a^{q-p-1}\alpha(p-1, p, r-1) + a^{q-p-2}\alpha(p-1, p+1, r-1) \\
& \qquad\qquad\qquad + \ldots + \alpha(p-1, q-1, r-1),
\end{aligned} \tag{3}$$

the first and last symbols remaining constant for each row, and in the columns,

the last symbol increasing from q to $r-1$, and then the first symbol increasing from 0 to $p-1$. For example, $\alpha(0, 3, 5, 7) = Q + abcd.\alpha(0, 2, 4, 5)$, where

$$\begin{aligned} Q = {} & a^4\alpha(0, 3, 5) + a^3\alpha(0, 4, 5) \\ & + a^3\alpha(0, 3, 6) + a^2\alpha(0, 4, 6) \\ & + a^2\alpha(1, 3, 6) + a\alpha(1, 4, 6) \\ & + a\alpha(2, 3, 6) + \alpha(2, 4, 6). \end{aligned}$$

Accordingly, by the rule for developing the co-factors of the third order, we have

$$\begin{aligned} Q = {} & a^4[\Sigma b^3c^2 + \Sigma b^3cd + 2\Sigma b^2c^2d] \\ & + a^3[\Sigma b^4c^2 + \Sigma b^4cd + 2\Sigma b^3c^3 + 3\Sigma b^3c^2d + 4\Sigma b^2c^2d^2] \\ & + a^2[\Sigma b^4c^3 + 2\Sigma b^4c^2d + 3\Sigma b^3c^3d + 4\Sigma b^3c^2d^2] \\ & + a\ [\Sigma b^4c^3d + 2\Sigma b^4c^2d^2 + 3\Sigma b^3c^3d^2] \\ & + \Sigma b^4c^3d^2 + 2\Sigma b^3c^3d^3, \end{aligned}$$

which is readily seen to be the symmetric function

$$Q = \Sigma a^4b^3c^2 + \Sigma a^4b^3cd + 2\Sigma a^4b^2c^2d + 2\Sigma a^3b^3c^3 + 3\Sigma a^3b^3c^2d + 4\Sigma a^3b^2c^2d^2.$$

5. It is obvious that, in the example above, the coefficients of those Σ's whose typical terms contain a^4 might have been determined solely by means of the first term in Q, viz., $a^4\alpha(0, 3, 5)$; and in like manner, the coefficients of the remaining Σ's, all of whose typical terms contain a^3, might have been determined from the corresponding terms in the value of Q. Thus, if we first write out the series of simple symmetric functions which enter the expression, beginning with $\Sigma a^4b^3c^2$, then for the purpose of determining the coefficients it is only necessary to consider a portion of Q, viz.,

$$\begin{aligned} Q = {} & a^4\alpha(0, 3, 5) + a^3\alpha(0, 4, 5) \\ & + a^3\alpha(0, 3, 6) + \text{etc.} \\ & + \text{etc.}; \end{aligned}$$

and by means of each term in this reduced expression we may write out a group of coefficients, depending mainly upon the exponents of b and d, in accordance with the rule given in §6 of the preceding article, but with the restriction mentioned therein with respect to the highest admissible coefficient. The process is as follows:

$\Sigma a^4b^3c^2$,	Σa^4b^3cd,	$\Sigma a^4b^2c^2d$,	$\Sigma a^3b^3c^3$,	$\Sigma a^3b^3c^2d$,	$\Sigma a^3b^2c^2d^2$
1	1	2	1	1	1
			1	2	3
1	1	2	2	3	4

$$\therefore\ Q = \Sigma a^4b^3c^2 + \Sigma a^4b^3cd + 2\Sigma a^4b^2c^2d + 2\Sigma a^3b^3c^3 + 3\Sigma a^3b^3c^2d + 4\Sigma a^3b^2c^2d^2.$$

It is to be noticed that in some cases not all the coefficients given by the rule are written down. For example, in the case of the third term of Q, the first term of $\alpha(0, 3, 6)$ would be Σb^4c^2, but the term $a^3b^4c^2$ is not a typical term, being in fact a term included in the expression $\Sigma a^4b^3c^2$ whose coefficient, 1, has already been written. We therefore commence with the coefficient belonging to the term Σb^3c^3, which is unity, because the exponent of d is zero.

6. In general, the series of simple symmetric functions to be written begins with $\Sigma a^{r-3}b^{q-2}c^{p-1}$, and the power of a which occurs in the last term of the series is the lowest power which need be retained in the expression for Q. In other words, we may reject all powers of a whose exponents are less than $\frac{1}{4}(p+q+r-6)$.

In determining the coefficients corresponding to the term of Q containing $\alpha(u, v, w)$ when u is not zero, it is not necessary to reduce the term to the form $a^\theta(bcd)^u\alpha(0, v-u, w-u)$. The coefficient unity occurs when b has its highest exponent, or when d has its lowest exponent; in $\alpha(0, v-u, w-u)$ these are $w-u-2$ and zero respectively, which in $\alpha(u, v, w)$ become $w-2$ and u respectively, and are still the highest and lowest exponents. So also, these exponents not occurring, the coefficient is 2 if b has its next highest or d its next lowest exponent, and so on; but with the restriction that the highest admissible coefficient is the least of the quantities $w-u-(v-u)$ and $v-u$; that is $w-v$ and $v-u$, the differences of the symbols in $\alpha(u, v, w)$. Thus the rule is a simple extension of that given in §6 of the preceding article for the form $\alpha(0, p, q)$.

It is obvious, from the mode in which the rule was derived, that the least of the differences of the symbols is always the last coefficient of the group derived from $\alpha(u, v, w)$, so that this coefficient may be written at once without reference to the exponents of b and d.

7. By repeated application of formula (2) we should have

$$\alpha(0, p, q, r) = Q + abcd\,Q' + a^2b^2c^2d^2\,Q'' + \dots,$$

but for purposes of calculation it is better to write

$$\alpha(0, p, q, r) = Q_1 + Q_2 + \dots + Q_k, \tag{4}$$

retaining the factor $abcd$ in the expression for Q_2, so that Q_2 denotes the sum of the terms which in the rectangular block considered in §2 are adjacent to the terms included in Q_1. Thus Q_1 consists of those terms of the sum in formula (1) in which $u=0$ or $w=r-1$, Q_2 consists of those among the remaining terms in which $u=1$ or $w=r-2$, and so on. The number, k, of Q's is obviously the least of the numbers p and $r-q$, the height and depth of the rectangular block. In

calculating the value of $\alpha(0, p, q, r)$, we need, of course, retain in the expressions for Q_2, Q_3, etc. only the same powers of a that are retained in Q_1. For example, in calculating $\alpha(0, 4, 7, 9)$, $k = 2$, and, since a^4 is the lowest power of a that need be retained, we may write

$$
\begin{aligned}
Q_1 = {} & a^6\alpha(0, 4, 7) + a^5\alpha(0, 5, 7) + a^4\alpha(0, 6, 7) \\
& + a^5\alpha(0, 4, 8) + a^4\alpha(0, 5, 8) + \text{etc.} \\
& + a^4\alpha(1, 4, 8) + \text{etc.} \\
& + \text{etc.}, \\
Q_2 = {} & a^5\alpha(1, 4, 7) + a^4\alpha(1, 5, 7) + \text{etc.} \\
& + a^4\alpha(2, 4, 7) + \text{etc.} \\
& + \text{etc.}
\end{aligned}
$$

We may now write the coefficients determined by Q_2 immediately under those determined by Q_1; and, writing for abridgement only the exponents of b, c, and d in each group, the calculation is as follows:

$\Sigma a^6 \ldots$						$\Sigma a^5 \ldots$						$\Sigma a^4 \ldots$	
530,	521,	440,	431,	422,	332	540,	531,	522,	441,	432,	333	442,	433
1	1	1	2	2	3	1	1	1	2	2	2	1	1
						1	2	2	2	3	4	3	3
												2	3
							1	1	1	2	3	2	2
												1	2
1	1	1	2	2	3	2	4	4	5	7	9	9	11

8. If we remove the top line in the expression for Q_1 above, the terms of Q_2 may be derived from the remaining terms by simply transferring a unit from the third symbol of each α to the first symbol, the second symbol and the exponent of a remaining unchanged.

The reason is that we thus pass from the term $a^\theta\alpha(u, v, w)$ of Q_1 to the term $a^\theta\alpha(u + 1, v, w - 1)$, which, if it be a term of the rectangular block, is situated therein one place further forward and at the same time one place lower down, and is therefore a term of Q_2. The top line of Q_1 gives rise to no part of Q_2 in this mode of derivation, because it is situated in the front face of the rectangular block; in other words, the third symbol being q in this line, the rule would give an inadmissibly small value of w in formula (1). In like manner, if the whole value of Q_1 were written, the last line, in which the value of u is $p - 1$, would give rise to no part of Q_2 because the rule would give an inadmissibly large value of u in formula (1).

The same rule, of course, applies to the derivation of Q_3 from Q_2, and so on, when $k > 2$. We reject in each case the top line, and we reject the bottom line when the value of u would be too great, that is to say, greater than $p - 1$.

9. The simple relation which exists between the coefficients determined by $\alpha(u, v, w)$ and $\alpha(u+1, v, w-1)$ renders it unnecessary to write the expressions for Q_2, Q_3, etc. For, in $\alpha(u+1, v, w-1)$ the highest exponent of b is a unit less and the lowest exponent of d is a unit greater than the corresponding numbers in $\alpha(u, v, w)$, and moreover, each of the differences between consecutive symbols is less by a unit in the former than in the latter case; hence, whether in any given case the coefficient is determined by the exponent of b or of d, or by the limiting value of the coefficient, in accordance with the rule in §6, it is always a unit less when determined by $\alpha(u+1, v, w-1)$ than when determined by $\alpha(u, v, w)$. For example, all the coefficients determined by Q_2 in the calculation given in §7 might have been thus derived from the corresponding ones determined by Q_1, the upper line of coefficients given by Q_1 producing, for the reason given above, no derived coefficients.

10. If then, after writing the reduced expression for Q_1, we indicate for each line the number of Q's in which it or a corresponding line occurs, we may, as soon as we have written the group of coefficients determined by a term of Q_1, at once write the derived coefficients corresponding to the other Q's.

It follows at once, from the rule to reject the top lines in deriving the successive Q's, that the top line of Q_1 should be marked 1, the next line 2, and so on till we come to the k^{th} line. The subsequent lines are to be marked k unless the rule for rejecting the bottom line in forming the successive Q's applies. Now, since p is the value of v in the first column of the expression for Q, and since we must, as explained in §8, reject the bottom line when the first symbol u, as given by the rule, would be as great as p, it follows that the indicating number of a line cannot exceed the difference between the first two symbols or values of u and v in the first term of the line. This number is therefore to be taken as the indicating number when it is less than k.

11. The calculation of $\alpha(0, 5, 13, 17)$ is given below as an example. The first term is $\Sigma a^{14}b^{11}c^4$; the function being of the 29th degree, the lowest power of a to be retained is a^8, and the value of k is 4. The simple symmetric functions in the final development are indicated as in the preceding example, §7. It will be noticed that among those commencing with a^{13} none occur in which the exponent of b exceeds 12; and, in general, it follows from the mode in which Q is found that the sum of the exponents of a and b in any term cannot exceed their sum in the first term; in other words, the sum of the two highest exponents cannot exceed $r + q - 5$.

CALCULATION OF $\alpha(0, 5, 13, 17)$.

$$
\begin{array}{l|l}
1 & a^{14}\alpha(0,5,13)+a^{13}\alpha(0,6,13)+a^{12}\alpha(0,7,13)+a^{11}\alpha(0,8,13)+a^{10}\alpha(0,9,13)+a^{9}\alpha(0,10,13)+a^{8}\alpha(0,11,13)\\
2 & a^{13}\alpha(0,5,14)+a^{12}\alpha(0,6,14)+a^{11}\alpha(0,7,14)+a^{10}\alpha(0,8,14)+a^{9}\alpha(0,9,14)+a^{8}\alpha(0,10,14)\\
3 & a^{12}\alpha(0,5,15)+a^{11}\alpha(0,6,15)+a^{10}\alpha(0,7,15)+a^{9}\alpha(0,8,15)+a^{8}\alpha(0,9,15)\\
4 & a^{11}\alpha(0,5,16)+a^{10}\alpha(0,6,16)+a^{9}\alpha(0,7,16)+a^{8}\alpha(0,8,16)\\
4 & a^{10}\alpha(1,5,16)+a^{9}\alpha(1,6,16)+a^{8}\alpha(1,7,16)\\
3 & a^{9}\alpha(2,5,16)+a^{8}\alpha(2,6,16)\\
2 & a^{8}\alpha(3,5,16)+
\end{array}
$$

$\Sigma a^{14}\ldots$	
1140	1
1131	1
1122	1
1050	1
1041	2
1032	2
960	1
951	2
942	3
933	3
870	1
861	2
852	3
843	4
771	2
762	3
753	4
744	5
663	4
654	5
555	5

$\Sigma a^{13}\ldots$				
1240		1		1
1231		1		1
1222		1		1
1150	1	1		2
1141	1	2	1	4
1132	1	2	1	4
1060	1	1		2
1051	2	2	1	5
1042	2	3	2	7
1033	2	3	2	7
970	1	1		2
961	2	2	1	5
952	3	3	2	8
943	3	4	3	10
880	1	1		2
871	2	2	1	5
862	3	3	2	8
853	4	4	3	11
844	4	5	4	13
772	3	3	2	8
763	4	4	3	11
754	5	5	4	14
664	5	5	4	14
655	6	5	4	15

$\Sigma a^{12}\ldots$							
1250		1		1			2
1241		1		2	1		4
1232		1		2	1		4
1160	1	1		1			3
1151	1	2	1	2	1		7
1142	1	2	1	3	2	1	10
1133	1	2	1	3	2	1	10
1070	1	1		1			3
1061	2	2	1	2	1		8
1052	2	3	2	3	2	1	13
1043	2	3	2	4	3	2	16
980	1	1		1			3
971	2	2	1	2	1		8
962	3	3	2	3	2	1	14
953	3	4	3	4	3	2	19
944	3	4	3	5	4	3	22
881	2	2	1	2	1		8
872	3	3	2	3	2	1	14
863	4	4	3	4	3	2	20
854	4	5	4	5	4	3	25
773	4	4	3	4	3	2	20
764	5	5	4	5	4	3	26
755	5	6	5	5	4	3	28
665	6	6	5	5	4	3	29

$\Sigma a^{11}\ldots$											
1170	1	1		1			1				4
1161	1	2	1	2	1		2	1			10
1152	1	2	1	3	2	1	3	2	1		16
1143	1	2	1	3	2	1	4	3	2	1	20
1080	1	1		1			1				4
1071	2	2	1	2	1		2	1			11
1062	2	3	2	3	2	1	3	2	1		19
1053	2	3	2	4	3	2	4	3	2	1	26
1044	2	3	2	4	3	2	5	4	3	2	30
990	1	1		1			1				4
981	2	2	1	2	1		2	1			11
972	3	3	2	3	2	1	3	2	1		20
963	3	4	3	4	3	2	4	3	2	1	29
954	3	4	3	5	4	3	5	4	3	2	36
882	3	3	2	3	2	1	3	2	1		20
873	4	4	3	4	3	2	4	3	2	1	30
864	4	5	4	5	4	3	5	4	3	2	39
855	4	5	4	6	5	4	5	4	3	2	42
774	5	5	4	5	4	3	5	4	3	2	40
765	5	6	5	6	5	4	5	4	3	2	45
666	5	7	6	6	5	4	5	4	3	2	47

$\Sigma a^{10} \ldots$

1090	1	1		1			1								4
1081	2	2	1	2	1		2	1			1				12
1072	2	3	2	3	2	1	3	2	1		2	1			22
1063	2	3	2	4	3	2	4	3	2	1	3	2	1		32
1054	2	3	2	4	3	2	5	4	3	2	4	3	2	1	40
991	2	2	1	2	1		2	1			1				12
982	3	3	2	3	2	1	3	2	1		2	1			23
973	3	4	3	4	3	2	4	3	2	1	3	2	1		35
964	3	4	3	5	4	3	5	4	3	2	4	3	2	1	46
955	3	4	3	5	4	3	6	5	4	3	4	3	2	1	50
883	4	4	3	4	3	2	4	3	2	1	3	2	1		36
874	4	5	4	5	4	3	5	4	3	2	4	3	2	1	49
865	4	5	4	6	5	4	6	5	4	3	4	3	2	1	56
775	4	6	5	6	5	4	6	5	4	3	4	3	2	1	58
766	4	6	5	7	6	5	6	5	4	3	4	3	2	1	61

$\Sigma a^{9} \ldots$

992	3	3	2	3	2	1	3	2	1		2	1			1			24
983	3	4	3	4	3	2	4	3	2	1	3	2	1		2	1		38
974	3	4	3	5	4	3	5	4	3	2	4	3	2	1	3	2	1	52
965	3	4	3	5	4	3	6	5	4	3	5	4	3	2	3	2	1	60
884	3	5	4	5	4	3	5	4	3	2	4	3	2	1	3	2	1	54
875	3	5	4	6	5	4	6	5	4	3	5	4	3	2	3	2	1	65
866	3	5	4	6	5	4	7	6	5	4	5	4	3	2	3	2	1	69
776	3	5	4	7	6	5	7	6	5	4	5	4	3	2	3	2	1	72

$\Sigma a^{8} \ldots$

885	2	4	3	6	5	4	6	5	4	3	5	4	3	2	4	3	2	2	1	68
876	2	4	3	6	5	4	7	6	5	4	6	5	4	3	4	3	2	2	1	76
777	2	4	3	6	5	4	8	7	6	5	6	5	4	3	4	3	2	2	1	80

12. Mr. O. H. Mitchell has shown* that the whole number of terms in the co-factor of an alternant is the quotient obtained by dividing the difference product of the exponents by the difference product $\zeta^{\frac{1}{2}}(0, 1, 2 \ldots n-1)$. The number of terms in $\alpha(0, p, q, r)$ is, accordingly,

$$\tfrac{1}{12}\, pqr\,(q-p)(r-p)(r-q).$$

This result may be used as a verification of the calculated coefficients. Thus in the example above the whole number of terms should be 35,360. The single symmetric functions whose coefficients have been obtained represent 24, 12 or 4 terms each, according as the exponents are all different, two alike, or three alike. If then the sums of the coefficients of the terms of these several classes be

* *American Journal of Mathematics*, Vol. IV, p. 341 (December, 1881).

multiplied by 24, 12 and 4 respectively, the sum of the products should be 35,360; and this was found to be the case.

13. The following additional proof of Mr. Mitchell's result may be given: If in $\alpha(p, q, r, \ldots, z)$ we put $a = b = c = \ldots = l = 1$, the result will be equal to the number of terms in question since the value of each term will be unity. Hence this number is the value assumed by the quotient

$$\frac{A(p, q, r, \ldots, z)}{A(0, 1, 2, \ldots, n-1)}$$

under this hypothesis, which causes the quotient to assume the indeterminate form. Now if $\alpha, \beta, \gamma, \ldots, \lambda$ are the logarithms of $a, b, c, \ldots, l$, the alternant $A(p, q, r, \ldots, z)$ is the result of compounding the two arrays

$$\begin{array}{llll} 1, & \alpha, & \frac{\alpha^2}{2!}, & \frac{\alpha^3}{3!}, \ldots \\ 1, & \beta, & \frac{\beta^2}{2!}, & \frac{\beta^3}{3!}, \ldots \\ \vdots & \vdots & \vdots & \vdots \\ 1, & \lambda, & \frac{\lambda^2}{2!}, & \frac{\lambda^3}{3!}, \ldots \end{array} \qquad \text{and} \qquad \begin{array}{llll} 1, & p, & p^2, & p^3 \ldots \\ 1, & q, & q^2, & q^3 \ldots \\ \vdots & \vdots & \vdots & \vdots \\ 1, & z, & z^2, & z^3 \ldots \end{array}$$

Hence $A(p, q, \ldots, z)$ is the sum of all the products of the determinants which can be formed by selecting any n columns from the first array, each multiplied by the corresponding determinant formed from the second array. The term of lowest degree in $\alpha, \beta, \ldots, \lambda$ is that formed by selecting the first n columns of the first array, and the value of the corresponding determinant formed from the second array is $\zeta^{\frac{1}{2}}(p, q, \ldots, z)$.

When $a = b = \ldots = l = 1$, $\alpha, \beta, \ldots, \lambda$ vanish simultaneously, and the vanishing ratio of any two alternants is that of the terms of lowest degree in $\alpha, \beta, \ldots, \lambda$; that is, the vanishing value of the ratio

$$\frac{A(p, q, \ldots, z)}{A(p', q', \ldots, z')} \text{ is } \frac{\zeta^{\frac{1}{2}}(p, q, \ldots, z)}{\zeta^{\frac{1}{2}}(p', q', \ldots, z')},$$

and in particular the number of terms in $A(p, q, \ldots, z)$ is

$$\frac{\zeta^{\frac{1}{2}}(p, q, \ldots, z)}{\zeta^{\frac{1}{2}}(0, 1, \ldots, n-1)}.$$